MAT 201 - Notes

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12.3: The Dot Product

- $$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta$$
- Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.
- $$\boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2$$
- $$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\boldsymbol{v}$$

12.4: The Cross Product

- $$\boldsymbol{u} \times \boldsymbol{v} = (|\boldsymbol{u}||\boldsymbol{v}| \sin\theta)\boldsymbol{n}$$
- Nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel iff $\boldsymbol{u} \times \boldsymbol{v} = 0$.
- $$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$$
- $|\boldsymbol{u} \times \boldsymbol{v}|$ is the area of a parallelogram.
- $|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|$ is the volume of a parallelpiped.

12.5: Lines and Planes in Space

- A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\boldsymbol{v}$ is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v}$$

- The plane through $P_0(x_0, y_0, z_0)$ normal to $\boldsymbol{n} = A\hat{i} + B\hat{j} + C\hat{k}$ is

$$\boldsymbol{n} \cdot \vec{P_0 P} = 0 \textbf{ or } A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

  - Note: $\boldsymbol{n} = \langle A, B, C \rangle$

- Two planes are parallel iff their normals are parallel, or $\boldsymbol{n}_1 = k\boldsymbol{n}_2$ for some scalar k.

- The angle between planes is the angle between their normal vectors.

13.1: Curves in Space and Their Tangents

- Graph 3D curves by looking only at 2 variables at a time.

-
$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$$

- If $\boldsymbol{r}$ is a differentiable vector function and the length of $\boldsymbol{r}(t)$ is constant, then
$$\boldsymbol{r} \cdot \frac{d\boldsymbol{r}}{dt} = 0.$$

  - Note: You can prove this is true by taking the dot product of $\boldsymbol{r}(t) = c$ with itself.

13.2: Integrals of Vector Functions; Projectile Motion

-
$$\int \boldsymbol{r}(t)\, dt = \boldsymbol{R}(t) + \boldsymbol{C}$$

13.3: Arc Length in Space

- The length of a smooth curve $\boldsymbol{r}(t)$, $a \leq t \leq b$ traced as t increases from a to b is
$$L = \int_a^b |\boldsymbol{v}|\, dt.$$

- $$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\boldsymbol{v}(\tau)|\, d\tau$$

- $$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

14.1: Functions of Several Variables

- A point in a region R in the xy-plane is an **interior point** of R if a disk drawn around it is entirely in R.
- A point is a **boundary point** if a disk drawn around it lies outside of R and inside of R.
- A region is **open** if it consists entirely of interior points.
- A region is **closed** if it contains all of its boundary points.
- A region is **bounded** if it lies within a disk of finite radius.
- A region is **unbounded** if it is not bounded.

14.2: Limits

14.3: Partial Derivatives

- If partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), f is continuous at (x_0, y_0).
- If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a, b) and are continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

- If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

14.4: The Chain Rule

- $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

- If instead $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial y}{\partial r}$$

- Note that it's very similar for $\frac{\partial w}{\partial s}$

- Suppose $F(x,y)=0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, $\frac{dy}{dx} = \frac{-F_x}{F_y}$

- $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

14.5: Directional Derivatives and Gradient Vectors

- The **gradient** of $f(x,y)$ is the vector
$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

- If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$ then
$$D_{\boldsymbol{u}} f(P_0) = \nabla f(P_0) \cdot \boldsymbol{u}.$$
  - Note that $\boldsymbol{u}$ must be a unit vector.

- f increases most rapidly in the direction of the gradient vector ∇f at P. The directional derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f|$

- Similarly, f decreases most rapidly in the direction of $-\nabla f$, so $D_{\boldsymbol{u}} f = -|\nabla f|$

- Any direction $\boldsymbol{u}$ orthogonal to a gradient $\nabla f \neq 0$ leads to a directional derivative of 0.

- At every point (x_0,y_0) in the domain of a differentiable function $f(x,y)$, the ∇f is normal to the level curve through (x_0,y_0).

- $$\frac{d}{dt} f(\boldsymbol{r}(t)) = \nabla f(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)$$
  - Note this is very similar to the chain rule from 1D calc.

14.6: Tangent Planes and Differentials

- The **tangent plane** to the level surface $f(x,y,z)=c$ at a point $P_0 = (x_0,y_0,z_0)$ is
$$\nabla f(P_0) \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0.$$

- The **normal line** to the level surface at P_0 is
$$\langle x_0 + y_0 + z_0\rangle + \langle f_x(P_0), f_y(P_0) + f_z(P_0).$$

- The plane tangent to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

- Estimate the change in the value of a function f by moving a small distance ds from a point P_0 in the direction $\boldsymbol{u}$ by

$$df = (\nabla f(P_0) \cdot \boldsymbol{u})ds$$

- The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Let $f(x, y)$ have continuous first and second derivatives in some rectangle R. Let M be the upper bound for $|f_{xx}|, |f_{xy}|, |f_{yy}|$ in R. Then the **error** $E(x, y)$ of the linear approximation satisfies

$$|E(x, y)| = \frac{1}{2}M(|x - x_0| + |y - y_0|)^2$$

- If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the total differential of f is

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

14.7: Extreme Values, Saddle Points

- **First Derivative Test for Local Extrema:** If $f(x, y)$ has a local maximum or minimum at an interior point of (a, b) and if the first partial derivative exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
- An interior point where both f_x and f_y are zero or where both f_x and f_y and do not exist is a **critical point** of f.
  - Note: Every global max/min must be a local max/min. Every local max/min must be a critical point.
- **Second Derivative Test for Local Extrema:**
  - f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
  - f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
  - f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
  - The test is **inconclusive** if $f_{xx}f_{yy} - f_{xy}^2 = 0$.

14.8: Lagrange Multipliers

- Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z and λ that satisfy
$$\nabla f = \lambda \nabla g \text{ and } g(x, y, z) = 0.$$
- Remember to check the boundaries.

14.9: Taylor's Formula for Two Variables

- $$f(x, y) = f(0, 0) + x f_x + y f_y + \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})$$
- $$|E| \leq \frac{M}{3!}(|x - x_0| + |y - y_0|)^3$$

15.1: Double and Iterated Integrals over Rectangles

- **Fubini's Theorem:** If $f(x, y)$ is continuous throughout a region R, then $\int \int f(x, y) dx\, dy = \iint f(x, y) dy\, dx$.
  - Note that if R is rectangular, we can switch the bounds without redrawing the region R.

15.2: Double Integrals over General Regions

- Fubini's Theorem also applies just make sure you change your bounds of integration.
- *dy dx*: Sketch the region. Draw an arrow parallel to the y axis in the direction of +y. Where it enters is its lower bound. Where it leaves is its upper bound. The x-limits are the ones that include all vertical arrows you can draw through region R.
- *dx dy*: Same thing as above but draw arrows parallel to x axis instead.

15.3: Area by Double Integration

- The **area** of a closed, bounded region R is
$$A = \iint_R dA.$$

- The **average value** of f over R is

$$\frac{1}{\text{Area}} \iint_R f \, dA.$$

15.4: Double Integrals in Polar Form

- $$\iint_R f(r,\theta) \, dA = \iint f(r,\theta) \, r \, dr \, d\theta.$$

- Change of coordinates works the same as for $\oint$ 15.3.

15.5: Triple Integrals in Rectangular Coordinates

- $$V = \iiint_D dV$$

- The **average value** of F over D is

$$\frac{1}{\text{volume of D}} \iiint_D f \, dV$$

15.6: Applications of Triple Integrals

- $$M = \iiint_D \text{density}(x,y,z) \, dV$$

- Center of mass:

$$\overline{x} = \iiint_D x \, \text{density}(x,y,z) \, dV,$$

- $$\overline{y} = \iiint_D y \, \text{density}(x,y,z) \, dV$$

- $$\overline{z} = \iiint_D y \, \text{density}(x,y,z) \, dV$$

15.7: Triple Integrals in Cylindrical and Spherical Coordinates

- **Cylindrical Coordinates:**

$$x = r\cos\theta, \; y = r\sin\theta, \; z = z, \; \tan\theta = y/x$$

– Remember:

$$dV = r\ dz\ dr\ d\theta$$

and θ always comes from the x-axis

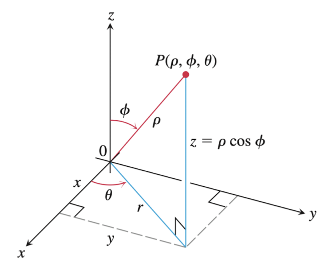
- **Spherical Coordinates:**

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$$

– Remember:

$$dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

and $0 \leq \phi \leq \pi$



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15.8: Substitutions in Mutliple Integrals

- The **Jacobian** of the coordinate transformation $x = g(u, v)$ and $y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

- Under the transformation $x = g(u, v)$ and $y = h(u, v)$,

$$\iint_R f(x, y) dx\ dy = \iint_G f(g(u, v), h(u, v)) \frac{\partial(x, y)}{\partial(u, v)} du\ dv$$

- For triple integrals, it's basically the same thing.

16.1: Line Integrals of Scalar Functions

- If f is defined on a curve C given parametrically by $\boldsymbol{r}(t) = \langle g(t), h(t), k(t) \rangle$, then the line integral is

$$\int_a^b f(g(t), h(t), k(t)) |\boldsymbol{v}(t)| dt$$

16.2: Vector Fields and Line Integrals: Work, Circulation, and Flux

- The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Evaluate the line integral of $F = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ along C: $\boldsymbol{r}(t) = \langle g(t), h(t), k(t)\rangle$:

  1. Substitute $x = g(t)$, $y = h(t)$, $x = k(t)$ into $M(x, y, z), N(x, y, z)$ and $P(x, y, z)$ of $\boldsymbol{F}$.
  2. Find the derivative velocity vector $\frac{dr}{dt}$.
  3. Evaluate
$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt} dt$$

- Line integrals with respect to dx, dy, dz:

  –
$$\int_C M\ dx + N\ dy + P\ dz,$$
    where $\int_C M(x, y, z)\ dx = \int_a^b M(g(t), h(t), k(t))\ g'(t)\ dt$, and so on for N and P.

- The **flow** along the curve is just the line integral along that curve.

- The **flux** of a vector field $\boldsymbol{F} = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$ if $\boldsymbol{n}$ is the outward pointing normal vector is
$$\int_C \boldsymbol{F} \cdot \boldsymbol{n} = \oint_C M\ dy - N\ dx$$

  – If the motion is counterclockwise, $\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k}$. If it's clockwise, $\boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{T}$, where $\boldsymbol{k}$ is the unit vector in the z-direction and $\boldsymbol{T} = \frac{r'(t)}{|r'(t)|}$.

16.3: Path Independence, Conservative Fields, and Potential Functions

- If the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ along all paths C from A to B is the same, then the integral is **path independent** and the field $\boldsymbol{F}$ is **conservative**.

- If $\boldsymbol{F}$ is a vector field defined on open region D and $\boldsymbol{F} = \nabla f$ for some scalar function f on D, then f is a **potential function for F**.

- Line integrals in conservative fields:

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_A^B \nabla f \cdot d\boldsymbol{r} = f(B) - f(A)$$

- If $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ around every closed curve in D, the field $\boldsymbol{F}$ is conservative and vise versa.
- Let $\boldsymbol{F} = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$ be over an open, simply connected domain. Then, $\boldsymbol{F}$ is conservative iff

$$P_y = N_z, \quad M_z = P_x, \quad \text{and} \quad N_x = M_y.$$

- Maybe the thing about exact differential forms....?

16.4: Green's Theorem in the Plane

- The **circulation density** (curl $\boldsymbol{F} \cdot \boldsymbol{k}$) of a vector field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ at the point (x, y) is

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

- The **flux density** (**divergence**) of a vector field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ at the point (x, y) is

$$\text{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

- **Green's Theorem:**
  -
$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R (\text{curl } \boldsymbol{F} \cdot \boldsymbol{k})\, dx\, dy$$
  -
$$\oint_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \oint_C M\, dy - N\, dx = \iint_R (\text{div } \boldsymbol{F})\, dx\, dy$$

16.5: Surfaces and Areas

- A surface is parametrized by

$$\boldsymbol{r}(u, v) = f(u, v)\boldsymbol{i} + g(u, v)\boldsymbol{j} + h(u, v)\boldsymbol{k}$$

- The **area** of a smooth surface parametrized by $\boldsymbol{r}$, $a \le u \le b$, $c \le v \le d$ is

$$A = \iint_R |\boldsymbol{r}_u \times \boldsymbol{r}_v| dA = \int_c^d \int_a^b |\boldsymbol{r}_u \times \boldsymbol{r}_v| du\, dv$$

- The **surface area differential** is

$$d\sigma = |\boldsymbol{r}_u \times \boldsymbol{r}_v| du\, dv$$

  - In spherical coordinates, it's useful to remember $d\sigma = |\boldsymbol{r}_\phi \times r_\theta| = R^2 \sin\phi\, d\theta\, d\phi$

- The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \boldsymbol{p}|} dA,$$

where $\boldsymbol{p} = \boldsymbol{i}, \boldsymbol{j}$, or $\boldsymbol{k}$ is normal to R and $\nabla F \cdot \boldsymbol{p} \neq 0$.

- For a graph $z = f(x, y)$ over a region R in the xy plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

16.6: Surface Integrals

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \quad (2)$$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA, \quad (3)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{{f_x}^2 + {f_y}^2 + 1}\, dx\, dy. \quad (4)$$

-

- The **flux** (or **surface integral**) of a vector field $\boldsymbol{F}$ over a smooth surface S having chosen normal unit vectors $\boldsymbol{n}$ is

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma.$$

- The outward normal is given by

$$\hat{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}.$$

  So,

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \iint_R \boldsymbol{F} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, du\, dv$$

- If S is a part of a level surface $g(x, y, z) = c$, then

$$\boldsymbol{n} = \pm \frac{\nabla g}{|\nabla g|},$$

  so

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \iint_R \boldsymbol{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \boldsymbol{p}|} dA.$$

16.7: Stokes' Theorem

- $$\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F}$$

- **Stokes' Theorem:** Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$. Then, the circulation of $\boldsymbol{F}$ around boundary curve C in the counterclockwise direction with respect to the surface's unit normal vector $\boldsymbol{n}$ equals the integral of the curl vector field over S, a piecewise smooth surface:

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma$$

  - Right hand rule! Curl fingers in the direction of C counterclockwise and your thumb is the normal vector.

- $$\text{curl grad} = \boldsymbol{0} \quad \text{or} \quad \nabla \times \nabla f = \boldsymbol{0}$$

16.8: The Divergence Theorem

- The **divergence** of a vector field $\boldsymbol{F} = M(x, y, z)\boldsymbol{i} + N(x, y, z)\boldsymbol{j} + P(x, y, z)\boldsymbol{k}$ is

$$\text{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F}.$$

- The flux of vector field $\boldsymbol{F}$ across a piecewise smooth oriented closed surface S in the direction of the surface's outward unit normal $\boldsymbol{n}$ equals the triple integral of the divergence of $\boldsymbol{F}$ over the region D enclosed by the surface:

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \iiint_D \nabla \cdot \boldsymbol{F}\, dV$$

- For every vector field $\boldsymbol{F}$,

$$\mathrm{div}(\mathrm{curl}\boldsymbol{F}) = 0$$

Green's Theorem and Its Generalization to Three Dimensions

Tangential form of Green's Theorem: $$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

Stokes' Theorem: $$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$$

Normal form of Green's Theorem: $$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA$$

Divergence Theorem: $$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

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