MATH 275 Notes

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1 Introduction to Vectors

A **vector** is denoted by $\vec{v} = \langle 1, 6 \rangle$.

2 Introduction to Vector Operations

A **unit vector** is a vector with magnitude equal to 1: $|\hat{v}| = 1$.

3 Introduction to Dot & Cross Products

3.1 Dot Product

Given $\vec{v} = \langle v_1, v_2, v3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, we express the **dot product** of $\vec{v}$ and $\vec{w}$ as

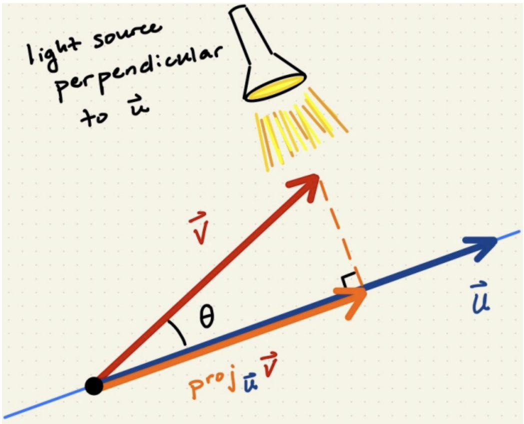
$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}||\vec{w}| \cos \theta.$$

3.2 Cross Product

$$\begin{aligned} \vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2 w_3 - v_3 w_2)\hat{i} - (v_1 w_3 - v_3 w_1)\hat{j} + (v_1 w_2 - v_2 w_1) \end{aligned}$$

The vector formed by $\vec{v} \times \vec{w}$ is orthogonal to the plane spanned by $\vec{v}$ and $\vec{w}$. The direction is determined by the right-hand rule.

3.3 Projections



A **vector projection** $\text{proj}_{\vec{u}}\vec{v}$ is the "shadow" of a vector $\vec{v}$ on the line spanned by a non-zero vector $\vec{u}$, cast by a light source whose rays are perpendicular to the second vector.

The **scalar component** of a projection is denoted by

$$\text{comp}_{\vec{u}}\vec{v} = |\vec{v}| \cos\theta = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}.$$

4 Equations of Lines and Planes

The **vector equation of a line** is

$$\vec{r}(t) = t\vec{v} + \vec{r}_0,$$

where $\vec{r}(t)$ is the position function, $\vec{v}$ is a vector parallel to the line, $\vec{r}_0$ is the position vector of a fixed point on the line, and t is the parameter.

The **vector equation of a plane** is

$$\vec{n} \cdot \overrightarrow{P_0P} = 0,$$

where $\vec{n}$ is a **normal vector** to the plane, $P_0 = (x_0, y_0, z_0)$ is an arbitrary fixed point on the plane, and $P = (x, y, z)$ is an arbitrary point in $\mathbb{R}^3$.

5 Vector Functions & Curves

A **vector function**, denoted $\vec{r}(t)$, is a function whose input is a paramter t and output is a vector $\vec{r}(t)$. For a 3-D vector function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

the **coordinate functions** are $x(t), y(t)$ and $z(t)$.

6 Derivatives of Vector Functions

A **vector derivative**, denoted $\vec{r}'(t)$ or $\frac{d\vec{r}}{dt}$, is computed by differentiating the components of $\vec{r}(t)$:

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **angle of intersection** between two curves is the angle between their tangent vectors at that point.

The **unit tangent vector**, denoted $\hat{T}$, is a tangent vector that has unit length: $|\hat{T}| = 1$.

A **vector differential** is an infinitesimal displacement vector: $d\vec{r} = \vec{r}'(t)dt$.

7 Topic 2.4

7.1 Gradient

The **gradient** of a multivariable function is computed by

1. Computing the partial derivatives of the function
2. Writing the vector field whose components are the corresponding partial derivatives

$$\vec{\nabla} f(x, y, z) = \langle f_x, f_y, f_z \rangle.$$

The gradient of a function is orthogonal to level sets (contour lines) and indicate the direction in which the function increases most rapidly.

7.2 Directional Derivatives

The directional derivative of a function f indicated by a unit vector $\vec{u}$ is

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}.$$

7.3 Relationships Between Gradient and Directional Derivatives

$$\begin{aligned} D_{\hat{u}} f &= \vec{\nabla} f(P) \cdot \hat{u} \\ &= |\vec{\nabla} f(P)|(1) \cos\theta \\ &= |\vec{\nabla} f(P)| \cos\theta \end{aligned}$$

8 Optimizations Subject to Constraint - Lagrange Multiplier Method

The **Lagrange multiplier**, denoted λ, is the parameter in the equation

$$\vec{\nabla} f = \lambda \vec{\nabla} g,$$

where f is the function to be optimized and g is the constraint.

9 Double Integrals

The standard double integral is

$$\iint_D f(x, y)\, dA,$$

where D is the region of integration, f is the function, and dA is an infinitesimally small area differential. For Cartesian coordinates, $dA = dx\, dy$.

Integrate from the inside outwards. Pretend the variable you're not integrating is a constant.

9.1 Finding Regions of Integration

Draw the region on a Cartesian plane. Draw an arrow parallel to the axis corresponding to the inner limit. The lower bound is where the arrow enters the region. The upper bound is where the arrow exits the region.

Project the region onto the other axis. That forms your lower and upper bounds for that variable.

9.2 Polar Coordinates

Recall the following:

- $r \geq 0$
- θ is measured counterclockwise from the x-axis
- $\omega \leq \theta \leq \omega + 2\pi$
- $x = r \cos \theta$
- $y = r \sin \theta$

With Cartesian coordinates, we have $dA = dx\, dy$. With Polar coordinates, we have $dA = r\, d\theta\, dr = r\, dr\, d\theta$.

10 Triples Integrals & Solids of Revolution

A **solid of revolution** is a 3-D region generated by spinning a 2-D planar region around an **axis of revolution**.

11 Scalar Line Integrals

Recall:

The vector line element $d\vec{r}$ is a vector that represents an infinitesimal displacement along a curve. The **scalar line element** ds is the magnitude of $d\vec{r}$: $ds = |d\vec{r}|$.

$$d\vec{r} = \vec{r}\,'(t)dt$$

So,

A **line integral** is an integral where the region of integration is a curve. A scalar line integral in $\mathbb{R}^3$ is

$$\int_c f(x, y, z)\, ds,$$

where c is the curve, f is the integrand, and ds is a scalar line element.

12 Vector Fields

A **vector field** is a function where the input is a point and the output is a vector. In $\mathbb{R}^3$,

$$\vec{F}(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k},$$

where P, Q, R are scalar-valued functions, or the **component functions** of the field.

13 Vector Line Integrals

Vector line integrals measure the interaction of a vector field $\vec{F}$ with an object moving through the field on the path c. These are very similar to scalar line integrals, except we use the expression

$$\int_c \vec{F} \cdot d\vec{r}.$$

To evaluate the integral we:

- Find limits of integration using the curve information given.
- Replace x, y, z in the integrand with $x(t), y(t)$, and $z(t)$ (from $\vec{r}(t)$).
- Compute $d\vec{r} = \vec{r}\,'(t)dt$.

14 Conservative Vector Fields

A **conservative vector field** is a gradient field. A vector field is conservative (meaning the vector field $\vec{F}$ is the gradient of the potential of the vector field $\vec{F} = \nabla f$) iff the partial derivatives of its component functions are equal.

In other words, we have the equation

$$\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}.$$

The vector field $\vec{F}$ is conservative iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

14.1 Finding the Potential Function

Drive $P(x, y)$ with respect to x, where the constant from integrating is a function of y. Do the opposite for $Q(x, y)$.

14.2 Properties of Line Integrals over Conservative Vector Fields

- Path independence
  - The line integrals of two curves with the same start and end points are the same.
- $$\int_c \vec{\nabla} f \cdot d\vec{r} = 0$$
  - c is closed
- **Fundamental Theorem of Line Integrals:** If $\vec{F} = \vec{\nabla} f$, then

  $$\int_c \vec{\nabla} f \cdot d\vec{r} = f(b) - f(a),$$

  where a and b are the start and end points, respectively, of the curve c.

15 Green's Theorem

Given that C is a closed (start and ends at the same point), simple (no self-intersection) curve oriented counter-clockwise in the xy-plane, D is the region enclosed by the curve, and $\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$, then

$$\int_C \vec{F}(x, y) \cdot d\vec{r} = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

16 Scalar Surface Integrals

Scalar surface integrals are similar to double integrals where the region of integration is 2-D, but with scalar surface integrals, the region of integration S is a 2-D surface floating in $\mathbb{R}^3$.

$$\int\int_S f(x,y,z)\,dS.$$

Integrate by

- Parameterize S: $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$
- Restrict $f(x,y,z)$ to S by replacing x with $x(u,v)$, y with $y(u,v)$, and so on.
- Compute $dS = |d\vec{S}| = |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|du\,dv$

Note that $dS = dA$ when the surface is parallel to one of the 2-D coordinate axes.

17 Vector Surface Integrals

$$\int\int_S \vec{F}(x,y,z) \cdot d\vec{S}.$$

Integrate by

- Parameterize S: $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$
- Restrict the vector field $\vec{F}(x,y,t)$ to the surface S by replacing $\vec{F}(x,y,t)$ with $\vec{F}(\vec{r}(u,v))$
- Compute $d\vec{S} = \pm(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v})du\,dv$
  - Assign the positive or negative corresponding to the orientation given in the problem.

18 Curl & Divergence

The **nabla operator** is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

Given that we have a vector field $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$,

$$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}.$$

$$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}.$$

Curl starts with a vector field $\vec{F}$ and goes to another vector field curl($\vec{F}$). Divergence starts with a vector field $\vec{F}$ and ends with a scalar-valued function div($\vec{F}$).

If curl($\vec{F}$) $= \vec{0}$, we call the vector field curl free or irrotational. If div($\vec{F}$) $= 0$, we say the vector field is divergence free or incompressible.

18.1 Geometry of Curl & Divergence

Curl measures the local rotation of a vector field around a point. Divergence measures the local expansion/contraction of a vector field around a point.

19 Divergence Theorem

S is a closed surface oriented outwards, W is enclosed by S, and $\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$, then

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_W \text{div}(\vec{F})dV.$$