AMC 10/12 Formulas & Strategies

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Preface

Author & Motivation

Throughout my junior high/high school career, I've learned that the AMC's are hard. There's no way around it. "Genius" solvers tend to trivialize problems, and I always wondered how: How did you think of the solution? What prompted you to think of the solution? Am *I* even able to achieve things like this? The answer: experience.

It really does all boil down to experience. The more problems you do, the more insight you gain. You see enough problems that are similar to each other so that you begin to form connections. Often, "genius" solutions only take a change in perspective or mindset. That, however, is not easy to achieve–that's why we have competition math.

So, I wrote this summary as notes for myself and for you. This is heavily inspired by Sohil Rathi's The Book of Math Formulas and Strategies, but I decided to remove some rudimentary (and hard) ideas.[1] If these ideas aren't 'rudimentary' for you, I implore you to stay with competition math and work until even the formulas and strategies outlined here are trivial.

Math Beyond Competition

Competition math really is amazing, but you are limited to many concepts that stray away from 'higher-level' mathematics, such as college/grad-level courses. I encourage you to not only explore through competition math, but also dabble in higher-level math courses–they really are a pleasure. Math is not random formulas pumped from a genius's brain, but a set of concepts that are logically and proved. Often, these proofs are more satisfying than their rote memorization, formula counterparts.

[1] I've decided that the 'hard ideas' aren't necessary to do well on either of these exams, but as AMC's problems become harder and harder, they may be necessary.

1 Algebra

1.1 Mean, Median, and Mode

- $$\text{Mean} = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$
- $$\text{Mode} = \text{Most Frequent Number}.$$
- $$\text{Median} = \text{Middle Term}.$$
  - If number of terms is even, Median = Average of Middle terms.
  - To find which term is the median, take the total terms, add one, and divide by two.[2]

1.2 Arithmetic Sequences

- Given the arithmetic sequence $a, a+d, a+2d, \ldots$:
  - $$a_n = a_1 + (n-1)d.$$
  - $$n = \frac{a_n - a_1}{d} + 1.$$
  - $$\text{Average} = \frac{a_1 + a_n}{2} = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$
  - $$S = \frac{a_1 + a_n}{2} \cdot n = \frac{2a_1 + (n-1)d}{2} \cdot n.$$

1.3 Geometric Sequences

- Given the geometric sequence $a, ar, ar^2, \ldots$:
  - $$a + ar + ar^2 + \ldots = \frac{a}{1-r}$$
  - $$a + ar + ar^2 + \ldots + ar^n = a\frac{1-r^{n+1}}{1-r}.$$

[2] If the number terms in the sequence is even, then you take the floor and ceiling of the decimal you get.

1.4 Special Series

- $$1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$
- $$1+3+5+\ldots+(2n-1)=n^2.$$
- $$2+4+6+\ldots+2n=n(n+1).$$
- $$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$
- $$1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

1.5 Algebraic Manipulations

1.5.1 Quadratic Factorizations

$$\begin{aligned}
x^2-y^2 &= (x-y)(x+y)\\
(x+y)^2 &= x^2+2xy+y^2=(x-y)^2+4xy\\
(x-y)^2 &= x^2-2xy+y^2=(x+y)^2-4xy\\
(x+y)^2+(x-y)^2 &= 2(x^2+y)^2\\
(x+y)^2-(x-y)^2 &= 4xy\\
(x+y+z)^2 &= x^2+y^2+z^2+2(xy+xz+yz)
\end{aligned}$$

1.5.2 Cubic Factorizations

$$\begin{aligned}
x^3+y^3 &= (x+y)(x^2-xy+y^2)\\
x^3-y^3 &= (x-y)(x^2+xy+y^2)\\
(x+y)^3 &= x^3+3xy(x+y)+y^3\\
(x-y)^3 &= x^3-3xy(x-y)+y^3\\
x^3+y^3+z^3-3xyz &= (x+y+z)(x^2+y^2+z^2-xy-xz-yz)
\end{aligned}$$

1.5.3 Sophie Germain's Identity

$$x^4+4y^4=(x^2-2xy+2y^2)(x^2+2xy+2y^2).$$

2 Counting & Probability

2.1 Definitions

2.1.1 Factorials and Arrangements

- The number of ways to arrange n objects in a line is given by $n!$.
- The number of ways to arrange n objects in a circle is $(n-1)!$, since you must divide by n rotations.

2.2 Combinatoric Strategies

2.2.1 Complementary Counting

Complementary counting is counting what we don't want and subtracting that from the total possible cases. "At least" in problem statements hints at using complementary counting.

2.2.2 Overcounting

Overcounting is when we count more than what we need and subtract the cases that we don't need to arrive at our answer. Note that we usually combine overcounting with PIE.

2.2.3 Casework

Many C&P problems can be solved using casework, which is solving a problem through dividing the general problem into cases and summing them at the end. Often, you should combine casework with other counting strategies. When looking for your cases, try and make cases easy to compute.

2.3 Advanced Concepts

2.3.1 Rearrangements and Counting

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- The number of rectangles of all sizes in a rectangular grid of size $m \times n$ is
$$\binom{m+1}{2}\binom{n+1}{2},$$
since two horizontal lines and two vertical lines create a unique rectangle.
- There are
$$\binom{n}{4}$$
ways to find an intersection point of two diagonals in an n-gon because we choose four points, of which there is only one way to form an intersection by connecting pairs of those four.

- We can get from $(0,0)$ to (x,y) in

$$\binom{x+y}{x} = \binom{x+y}{y}$$

ways.

  - Think of arranging u's and d's for ups and downs.

2.3.2 Stars and Bars

- We can distribute k indistinguishable objects into n distinguishable bins with the formula

$$\binom{n+k-1}{k}.$$

- When a problem has indistinguishable objects, we think of stars and bars.
- We can use stars and bars to solve linear equations like $a+b+c=4$, assuming that $a,b,c \geq 0$.
  - You can manipulate a counting problem into an equation problem.

2.3.3 Binomial Theorem

- For non-negative n, we have

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n.$$

  - Corollary:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

2.3.4 Combinatorial Identities

- Vandermonde's Identity:

$$\binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \ldots + \binom{n}{m}\binom{m}{0} = \binom{m+n}{n}.$$

- Pascal's Identity:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

- Hockey Stick Identity:

$$\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$$

2.3.5 Pigeonhole Principle

3 Probability and Expected Value

- $$\text{Probability} = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}}[3].$$

- The expected value of event X is

  $$\sum x_i \cdot P(x_i),$$

  where x_i denote the possible values of X and $P(x_i)$ denote the probability they occur.

- Linearity of Expectation (regardless if X is an independent or dependent event):

  $$E[x_1 + x_2 + \ldots + x_n] = E[x_1] + E[x_2] + \ldots + E[x_n].$$

3.0.1 Geometric Probability

We use geometric probability when there are an infinite or non-discrete number of cases. We denote total possible outcomes using lengths, areas, or volumes. Try to solve a geometric probability problem by

1. Try a couple examples for various cases, including testing boundary cases.
2. Generalize from these examples to try to find a region of successful outcomes.
3. Use geometry to find the length/area/volume of this region.

3.0.2 Principle of Inclusion-Exclusion (PIE)

PIE is a method we use to systematically overcount and correct for overcounting.

- $$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

- $$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

[3] Although this seems trivial, it's very important in more advanced applications of probability.

3.0.3 Bijections, Recursion, and States

States

We use states to try to find a probability of a "win" from different positions or turns:

1. Assign variables to the probabilities of winning from the different positions
2. Write your equations for the probability of winning from each of these positions in terms of variables and constants
3. Solve the system of equations

4 Number Theory

4.1 Primes

- To check if n is prime, we check all the primes that are less than or equal to $\sqrt{n}$.
- Every integer has a unique prime factorization of the form

$$p_1^{3_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{3_k},$$

which has

$$(e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$$

factors.

2	Last digit is even
3	Sum of digits is divisible by 3
4	Last 2 digits divisible by 4
5	Last digit is 0 or 5
6	Divisible by 2 and 3
7	Divide by 7 repeatedly
8	Last 3 digits are divisible by 8
9	Sum of digits is divisible by 9
10	Last digit is 0
11	Calculate the sum of odd digits (O) and even digits (E). If $\lvert O - E \rvert$ is divisible by 11, then the number is also divisible by 11
12	Divisible by 3 and 4
15	Divisible by 3 and 5

- We count the number of factors of prime p in $n!$ with the following formula:

$$\left\lfloor \frac{n}{p^1} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots.$$

4.2 GCD & LCM

- GCD is found by taking the lowest exponents of the prime factorizations of m and n.
- LCM is found by taking the highest exponents of the prime factorizations of m and n.
- GCD of two numbers must be a factor of the LCM.
- $$\gcd(m, n) \cdot \operatorname{lcm}(m, n) = mn.$$
- $$\gcd(ac, bc) = c \cdot \gcd(a, b).$$
- When you draw a diagonal of a rectangular grid, the number of unit squares it will pass through is $a + b - \gcd(a, b)$, where a and b are the grid's side lengths.
- Euclidean Algorithm:
  $$\gcd(x, y) = \gcd(x, y - kx).$$
- Euler's Totient Function: If a number n has the prime factorization
  $$p_1^{e_1} \cdot p_2^{e_2} \dots p_n^{e_n},$$
  then
  $$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_n}\right),$$
  where $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n.
- Euler's Totient Theorem:
  $$a^{\phi(n)} \equiv 1 \pmod{n}$$
  iff
  $$\gcd(a, n) = 1.$$

4.3 Modular Arithmetic

- If $a \equiv x \pmod{n}$ and $b \equiv y \pmod{n}$, then
  $$ab \equiv xy \pmod{n}.$$
- If $a \equiv x \pmod{n}$, then
  $$a^m \equiv x^m \pmod{n}.$$

5 Geometry

6 Trigonometry

7 Logarithms

8 Complex Numbers

9 General Tips

- Use tables and Venn Diagrams.