Introduction to Geometry (AoPS)

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1 Random Notes

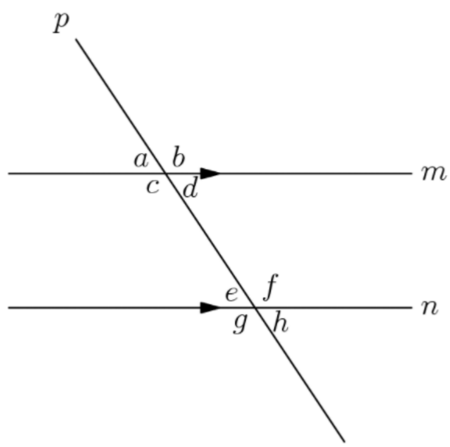
1.1 General

- Given a square of side length x, its area is x^2 or diagonal^2.
- A **locus** is a set of points defined by some property.
- Often, when it's not obvious how to prove an inequality directly, we find something that's between the two quantities we're comparing.
- **Midline theorem** or midsegment theorem says that cutting along the midline of a triangle creates a segment that is parallel to the base and half as long.
- Whenever you see convenient angles $(30, 45, 60, 90, 120$, etc.), try to build right triangles.

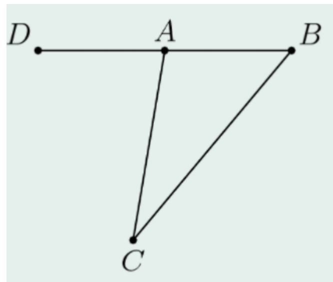
1.2 Constructions

- Use circles and lines to construct things.

2 Angles and Congruent Triangles



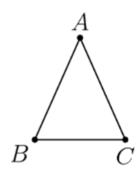
- Line p is called a transversal. If m and n are parallel, then $a = d = e = h$ and $b = c = f = g$ by properties of corresponding angles.
- If $a = e$, $d = e$, etc., then m and n must be parallel.



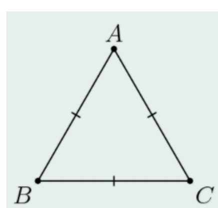
- We know that $\angle B + \angle C = \angle DAC$.
- Triangles are congruent by ASA, SAA, AAS, SAS, and SSS

3 Isosceles & Equilateral Triangles, Perimeter, and Area

3.1 Isosceles & Equilateral Triangles



- If $AB = AC$, then $\angle B = \angle C$. If $\angle B = \angle C$, then $AB = AC$. We call $\triangle ABC$ an **isosceles triangle**.

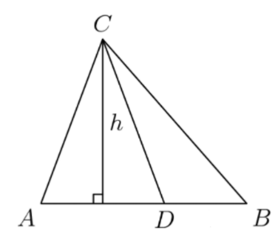


- An **equilateral triangle** is an isosceles triangle in which all three sides are equal and all three angles are $60°$.

3.2 Perimeter & Area

- The **perimeter** of a figure is defined to be the total distance around the figure (the sum of the lengths of the segments that make up the boundary).

- The **area** of a figure tells us how much of a plane is contained inside a bounded figure, like a polygon.
- Area equations:
  - Rectangle: lw or bh.
  - Triangle:
    1. $\frac{1}{2}bh$, where b is the base and h is the height.a[1]
    2. Heron's Formula: $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter, and a, b, and c are lengths of a triangle.[2]
    3. rs, where r is the inradius and s is the semiperimeter.
    4. $\frac{1}{2}ab\sin C$, where a and b are the lengths that create $\angle C$.



- If two triangles share an altitude, then the ratio of their areas equals the ratio of their bases. In other words,

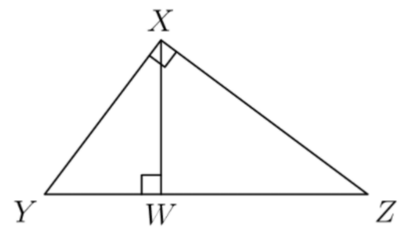
$$\frac{[ADC]}{[BDC]} = \frac{AD}{BD}, \frac{[ADC]}{[ABC]} = \frac{AD}{AB}.$$

- "Complementary counting" for area is cool.

4 Similar Triangles

- **Similar triangles** are triangles that have corresponding angles that are congruent and corresponding pairs of side lengths and altitudes that are all in the same ratio.
- AA, SAS, SSS, and HL (right triangles) proves similarity.[3]
- If a problem involves ratios of lengths, chasing angles, or parallel lines, think of similar triangles (you may need to construct a line).

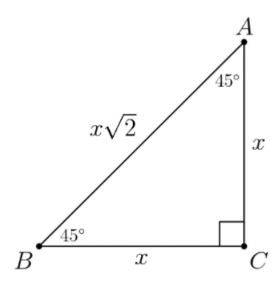
[1]Altitude can be used interchangeably for height.
[2]Look for the proof in the "Proof" section.
[3]If you know two angles of a triangle, you know the third. So, AA similarity is identical to AAA similarity.



- Often, we can draw an altitude from the vertex of a right angle to create two similar triangles.
- Use common ratios between similar triangles.
- When given lengths, look to use SAS or SSS.
- Given any pair of similar triangles, the ratio of their areas equals the square of the ratio of their corresponding sides (or altitudes).

5 Right Triangles

- **Pythagorean Theorem**: If you have a right triangle with legs a and b and hypotenuse c, then $a^2 + b^2 = c^2$. If $a^2 + b^2 = c^2$, then the triangle is right.
- A **Pythagorean triple** is a set of positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$.
  - $\{3, 4, 5\}, \{5, 12, 13\}, \{7, 24, 25\}, \{8, 15, 17\}$.
- Build a right triangle in which the length you want to find is one of the sides.
- If $a^2 + b^2 > c^2$, then acute. If $a^2 + b^2 = c^2$, then right. If $a^2 + b^2 < c^2$, then obtuse.



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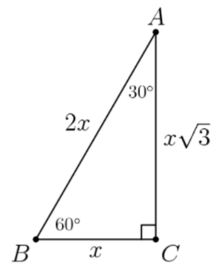

A
30°
2x
x√3
60°
B
x
C

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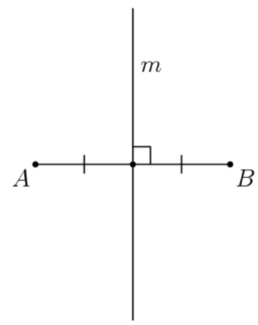
6 Special Parts of a Triangle

6.1 Perpendicular Bisector

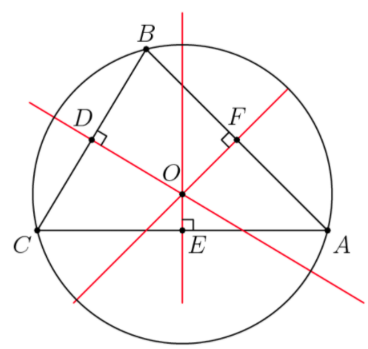
- The **perpendicular bisector** of a line segment is the line that is perpendicular to the segment and goes through its midpoint.



- A point is equidistant from A and B if and only if it lies on the perpendicular bisector of $\overline{AB}$.[4]

6.2 Circumcenter

- The **circumcenter** is the intersection of all the perpendicular bisectors of a triangle, typically labeled O.

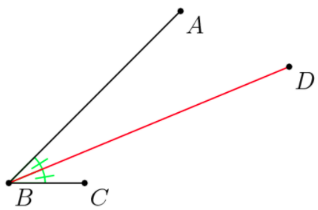


- The circumcircle of $\triangle ABC$ the unique circle passing through points A, B, and C.
  - Thus, it is equidistant from all vertices of $\triangle ABC$.

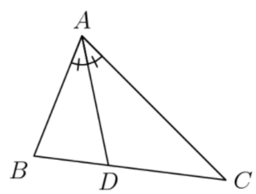
[4]If a statement and its converse is true, we use the phrase "if and only if" (abbreviated as "iff").

6.3 Angle Bisectors

- The **angle bisector** is the line that splits an angle into two congruent angles.



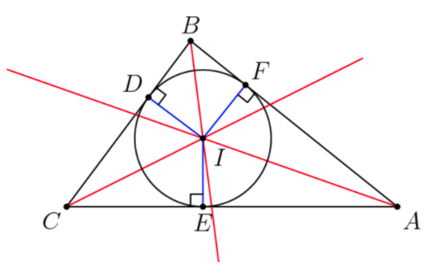
- All points on $\overrightarrow{BD}$ are equidistant from $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
- **Angle Bisector Theorem**:



For $\triangle ABC$ with angle bisector $\overline{AD}$,

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

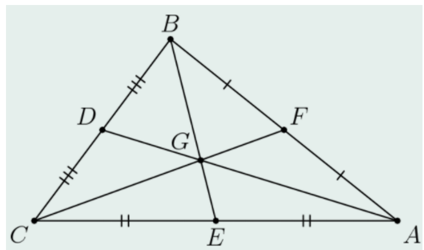
6.4 Incenter



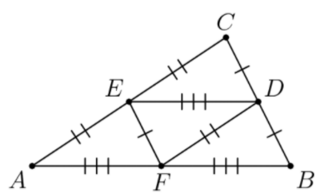
- The **incenter** is the intersection of the angle bisectors of a triangle, typically labeled I.
-
$$[ABC] = rs,$$
where r is the inradius and s is the semiperimeter.

6.5 Medians

- A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.



- The medians split $\triangle ABC$ into six triangles with equal area.
- $\triangle DEF$ is called the **medial triangle** of $\triangle ABC$.[5]



  - $\triangle ABC \sim \triangle AFE \sim \triangle EDC \sim \triangle FBD \sim \triangle DEF$.
  - $\overline{DF} \parallel \overline{AC}, \overline{ED} \parallel \overline{AB}, \overline{EF} \parallel \overline{BC}$.
  - $ED = \frac{1}{2}AB$, and the same ratio applies with EF and DF to their respective segments.
  - EF is equidistant from point A as $\overline{BC}$.
- The median to the hypotenuse of a right triangle is half the hypotenuse.

6.6 Centroid

- The **centroid** is the intersection of the three medians of a triangle, typically labeled G.
- The centroid splits each median into a $2:1$ ratio.
- The centroid is the average of their respective x-y coordinates on a Cartesian plane.

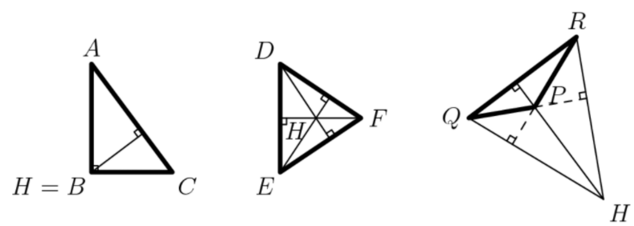
[5] because you connect $\triangle ABC$'s midpoints

6.7 Altitudes

- **Altitudes** basically mean the same thing as height.

6.8 Orthocenter

- The **orthocenter** is the intersection of the three altitudes of a triangle, typically labeled H.
- The orthocenter of a right triangle, acute triangle, and obtuse triangle are at the right angle, inside the triangle, and outside the triangle, respectively.

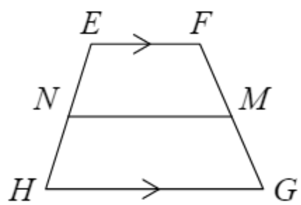


7 Quadrilaterals

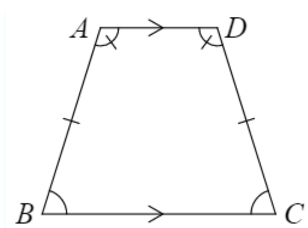
7.1 Quadrilateral Basics

- **Convex quadrilaterals** are quadrilaterals such that all interior angles are less than 180°.
- The interior angles of any quadrilateral add to 360°.

7.2 Trapezoids



- In any **trapezoid**:
  - $\overline{EF} \parallel \overline{HG}$.
  - $NM = \frac{EF+HG}{2}$
  - $[EFGH] = \frac{EF+HG}{2}h = (NM)h$.[6]
  - $[EFGH] = \frac{1}{2}\text{diagonal}_1 \cdot \text{diagonal}_2$
  - The distance from $\overline{EF}$ to $\overline{NM}$ is equal to the distance from $\overline{NM}$ to $\overline{HG}$.



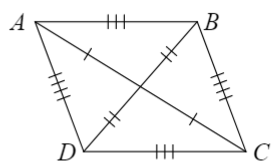
- In an **isosceles trapezoid**:[7]
  - Base angles come in two equal pairs.
  - Legs are equal.
  - Diagonals are equal.

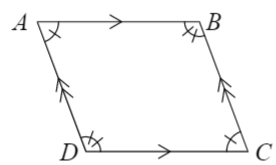
[6]This can also be read as the area of a trapezoid is equal to its median times its height.
[7]An isosceles trapezoid is defined as having two base angles which are equal.

If any of these are true, the others must be true.

- Draw altitudes to simplify trapezoid problems.

7.3 Parallelograms



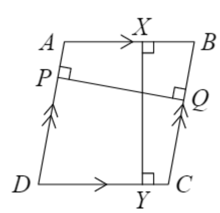


- In a parallelogram $ABCD$, the opposite sides are equal, the opposite angles are equal, and the diagonals bisect each other.

  Conversely, $ABCD$ is a parallelogram if *any one* of the following are true:

  1. $AB = CD$ and $AD = BC$ (opposite sides are equal).
  2. $\angle A = \angle C$ and $\angle B = \angle D$ (opposite angles are equal).
  3. Diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

  Alternatively, if you know opposite sides of a parallelogram are equal *and* parallel, then the quadrilateral is a parallelogram.



- The area of a parallelogram is its base times its height.

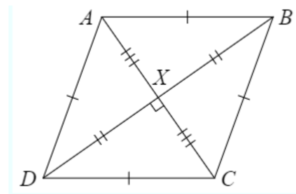
$$[ABCD] = (AB)(XY) = (BC)(PQ).$$

- The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides of the parallelogram.

7.4 Rhombi

- A **Rhombus** is a quadrilateral in which all four sides are congruent.
- A Rhombus is a parallelogram.

- A rhombus' diagonals are perpendicular to each other.[8]



- The area of a rhombus is half of the product of its diagonals (or base times height).

$$[ABCD] = \frac{1}{2}(AC)(BD).^{9}$$

7.5 Rectangles

- A **rectangle** is a quadrilateral in which all the sides are equal.
- A rectangle's diagonals are congruent.
- All properties of parallelograms apply to rectangles.[10]

7.6 Squares

- A **square** is a quadrilateral with equal angles and sides.
- Everything true about rectangles, rhombi, and parallelograms are true about squares.

7.7 If and Only If

- If and only if is commonly used in quadrilateral classifications, so it's in this section (and because that's how it is in the book).
  - If an only if is abbreviated as iff. Another way of saying iff is "necessary and sufficient" or $\Leftrightarrow$.
- When proving iff statements, we need to separately prove the if and only if portions.
  - In other words, we have to prove a statement (if) and its converse (only if).

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[8]The proof for this is trivial, and thus not in the proof section. Parallelogram diagonals bisect each other. Through SSS, we prove that each of the four triangles created by the diagonals are congruent to each other. Thus, rhombus' diagonals are perpendicular to each other.

[9]Again, this proof is trivial. Create the diagonals of the rhombus and realize that the four triangles created are congruent right triangles.

[10]But not the other way around.

- Assuming the problem says A iff B:
  - We have to prove that A if B, or in other words, given B, prove A is true.
  - We have to prove that A only if B, or in other words, if A is true, prove B is true.

8 Polygons

8.1 Introduction to Polygons

- **Polygons** are closed planar figures with line segments as boundaries.
- **Regular polygons** are polygons in which the sides are equal and all angles are equal.

8.2 Angles in a Polygon

- The sum of the interior angles in an n-sided polygon is $180(n-2)$. Thus, each individual angle has measure $180(n-2)/n$.
- The sum of the exterior angles in a convex polygon with n sides is 360 (no matter the polygon). Thus, the measure of each exterior angle in a regular n-gon is $360/n$.
  - Sometimes, using the exterior angles is easier, since they always sum to 360 (less solving).

8.3 Polygon Area

- Split polygons into easier areas, like triangles, parallelograms, trapezoids, etc. If that doesn't work, think of extending some lines to create a new polygon and subtracting areas.
- The area of a regular hexagon is

$$\frac{6s^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{2},$$

  where s is the side length of a the hexagon.
- The area of a regular polygon is half its perimeter times the distance from the center of the polygon to a side (its apothem).
- You can find the area of an octagon by extending its sides to form a square (because its interior angles are 135).

8.4 Polygon Problems

- There are $n(n-3)/2$ diagonals in an n-gon.

9 Geometric Inequalities

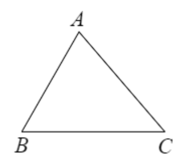
9.1 Sides and Angles of a Triangle

- In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. The middle side, of course, is therefore opposite the middle angle.

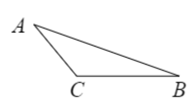
  In other words, in $\triangle ABC$, $AB \geq AC \geq BC$ iff $\angle C \geq \angle B \geq \angle A$.

9.2 Pythagoras–Not Just for Right Triangles?

- $\angle C$ of $\triangle ABC$ is acute iff $AB^2 < AC^2 + BC^2$.



- $\angle C$ of $\triangle ABC$ is right iff $AB^2 = AC^2 + BC^2$.
- $\angle C$ of $\triangle ABC$ is obtuse iff $AB^2 > AC^2 + BC^2$.



9.3 The Triangle Inequality

- The **Triangle Inequality** states for any three points A, B, and C, we have

$$AB + BC > AC.$$

  $AB + BC = AC$ when $\triangle ABC$ is a **degenerate triangle**, or the vertices are colinear.

  – A natural corollary is that any side of a triangle must be less than half the triangle's perimeter.

10 Introduction to Circles

10.1 Arc Measure, Arc Length, and Circumference

- An **arc** is the portion of a circle connecting two points on the circle's circumference.

- A circle's **circumference** is the perimeter of the circle.

$$C = \pi d = 2\pi r.$$

- A **central angle** of a circle is an angle with the center of the circle as its vertex.
- Two chords of a circle are congruent iff the arcs they subtend are congruent.
- Connect key points to the center of the circle.
- The area of a circle is given by

$$\pi r^2.$$

- A radius bisects a chord if the radius is perpendicular to the chord. The converse, namely that a radius is perpendicular to a chord if the radius bisects the chord, is also true.

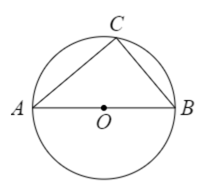
10.2 Funky Areas

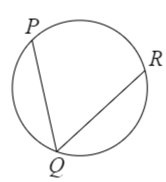
- Nearly all funky area problems are solved by expressing the funky area as sums and/or differences of pieces whose areas we can easily find. The first step should be clearly expressing the funky area in terms of simple areas.
- In problems involving multiple circles, connecting the centers can be helpful. In problems involving intersecting circles, connecting the intersection points to the centers (and to each other) is often useful.

11 Circles and Angles

11.1 Inscribed Angles

- **Thales Theorem** says that any angle inscribed in a semicircle is a right angle.

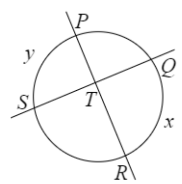




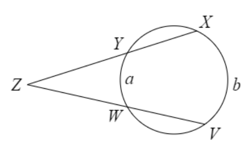
- The measure of an inscribed angle is one-half the measure of the arc it intercepts, or
$$\angle PQR = \frac{\overset{\frown}{\text{PR}}}{2}.$$
- Any two angles that subtend the same arc are equal.

11.2 Angles Inside and Outside Circles

- A line that intersects in two points is called a **secant**.



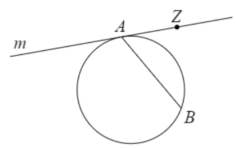
- The measure of the angle formed by two intersecting chords is the average of the measures of the arcs intersected by the chords.
$$\angle PTS = \angle QTR = \frac{\overset{\frown}{\text{PR}} + \overset{\frown}{\text{QR}}}{2} = \frac{x+y}{2}.$$



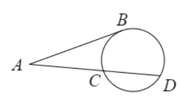
- Two secants that meet at a point outside a circle form an angle equal to half the difference of the arcs they intercept.
$$\angle Z = \frac{b-a}{2}.$$

11.3 Tangents

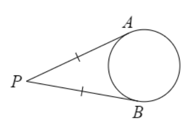
- A line is tangent iff a line passes through a point on the circle such that it is perpendicular to the radius drawn to that point.
- An angle formed by a tangent and a chord with the point of tangency as an endpoint equals one-half the arc intercepted by the angle.
$$\angle ZAB = \frac{\overset{\frown}{\text{AB}}}{2}.$$



- The angle formed by a tangent and a secant is half the difference of the intercepted arcs.
$$\angle BAD = \frac{\overset{\frown}{BD} - \overset{\frown}{BC}}{2}.$$

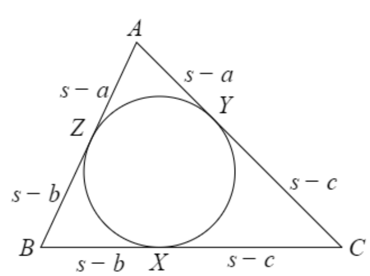


- Two tangents that meet at the same point are equal in length and the angle created by them is equal to half of the difference of the major and minor arc they create.
$$AP = BP, \angle APB = \frac{\text{major } \overset{\frown}{AB} - \text{minor } \overset{\frown}{AB}}{2}.$$



11.4 General Problems

- A **cyclic quadrilateral** is a quadrilateral that is inscribed in a circle.
  - Opposite angles sum to 180°.
- Draw radii to points of tangency and connect circles' centers.
- Always look for right triangles.



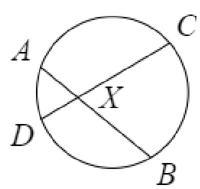
- $$AZ = AY = s - a$$
$$BZ = BX = s - b$$
$$CX = CY = s - c$$

  where $AB = c, AC = b, BC = a$, and s is the semiperimeter.

12 Power of a Point

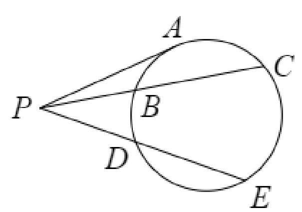
12.1 What is Power of a Point?

- Given a line through a point P that intersects a circle in two points, U and V, the **Power of a Point Theorem** states that for all such lines, $(PU)(PV)$ is constant.



- **Power of a point** gives us

$$(XA)(XB) = (XC)(XD).$$



- $$PA^2 = (PB)(PC) = (PD)(PE).^{11}$$

13 Three-Dimensional Geometry

13.1 Planes

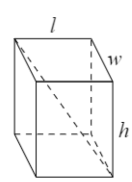
- A **plane** is a 'flat' two-dimensional surface that extends forever.

[11] A common mistake is to write Power of a Point as something like $(PB)(BC)$. The same point must appear in *all* segments in the equation.

- Any two or three points are **coplanar**, meaning there is a plane through all three of them. Not all four points are coplanar; we call these points **noncoplanar**.
- **Skew lines** are lines that are noncoplanar and do not intersect.[12]
- The angle between planes is called a **dihedral angle**.
- If a line segment is perpendicular to a plane, then every line in that plane must be perpendicular to the line segment.

13.2 Prisms

- **Polyhedrons** are 3-D shapes created by polygons as the bases and sides.
- A **prism** is a 3-D figure with two congruent parallel faces and parallelograms as the other faces.



- Given a rectangular prism of length l, width w, and height h:

$$\text{Volume} = lwh$$
$$\text{Surface Area} = 2(lw + wh + lh)$$
$$\text{Space Diagonal} = \sqrt{l^2 + w^2 + h^2}$$

- Given a cube with side length s:

$$\text{Volume} = s^3$$
$$\text{Surface Area} = 6s^2$$
$$\text{Space Diagonal} = s\sqrt{3}$$

- If each dimension is multiplied by a factor of k, then the surface area is multiplied by a factor of k^2 and volume increases by a factor of k^3.

[12] Parallel lines must be contained within the same plane, but never intersect.

13.3 Pyramids

- The **apex** of a pyramid is the point that isn't coplanar with the polygon base.
- A 'right' pyramid has its apex over the center of the base.
- $$\text{Volume of a Pyramid} = \frac{1}{3}Bh.$$
- A **tetrahedron** is a triangular pyramid.[13]
- $$\text{Volume of a Regular Tetrahedron} = \frac{s^3\sqrt{2}}{12}.$$

13.4 Regular Polyhedra

- A **regular polyhedron** a convex polyhedron in which all of the faces are congruent regular polygons, and there are the same number of edges at each vertex.
-

Name	Face Type	# Faces	# Edges	# Vertices
Tetrahedron	Triangle	4	6	4
Cube	Square	6	12	8
Octahedron	Triangle	8	12	6
Dodecahedron	Pentagon	12	30	20
Icosahedron	Triangle	20	30	12

- The **Euler characteristics** says that $V - E + F = 2$, for *any* polyhedra.

13.5 Cylinders

- Given a cylinder with height h and radius r has:

$$\text{Volume} = \pi r^2 h$$
$$\text{Lateral Surface Area} = 2\pi rh$$
$$\text{Total Surface Area} = 2\pi rh + 2\pi r^2$$

13.6 Cones

- Given a cone with radius r, height h, and slant height l:

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$
$$\text{Lateral Surface Area} = \pi rl$$
$$\text{Total Surface Area} = \pi rl + \pi r^2$$

[13] A regular tetrahedron is has all edges same length.

- The volume of a frustum is

$$\frac{1}{3}\pi(r_1^2 + r_1 r_2 + r_2^2)h.$$

13.7 Spheres

- A sphere of radius r has:

$$\text{Volume} = \frac{4\pi r^3}{3}$$
$$\text{Surface Area} = 4\pi r^2$$

- The radius of a sphere is perpendicular to any cross-section of the sphere.
- If a sphere is tangent to a plane, then the angle formed by the radius' intersection with the tangent point is a right angle.
- If two spheres are externally tangent, then the segment connecting their centers passes through the tangent point.
- If two spheres are internally tangent, the line passing through their centers passes through the tangent point.

13.8 General Ideas

- The ratio of the surface areas of similar 3-D figures is the square of the ratio of their corresponding side lengths. Moreover, the ratio of the volumes of similar 3-D figures is the cube of the ratio of their corresponding side lengths.
- With symmetrical systems of equations, try multiplying or adding all of them.
- Try including the center of spheres and tangent points in your cross-sections.
- Sometimes try "unfolding" or "unrolling" a 3-D figure to reveal a simpler 2-D figure.

14 Transformations

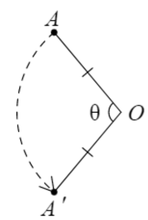
14.1 Translations

- A **translation** is a slide in a given direction.
- After transformations, we say that A' is the **image** of A and that A maps to A'.
- All *translations* create a figure congruent to the original.

- A **fixed point** of a transformation is a point that is its own image.
- The **identity** transformation is the transformation that maps every point to itself.

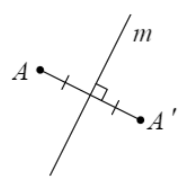
14.2 Rotations

- We **rotate** a figure by spinning it by some angle around some point.
- The image of point A under a **rotation** of angle θ counterclockwise about point O is the point A' such that $OA = OA'$ and $\angle AOA' = \theta$.

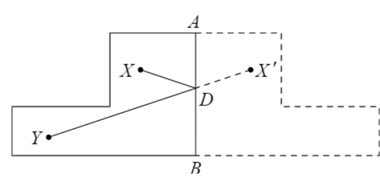


14.3 Reflections

- We can **reflect** point A over line m and get the point A' such that m is the perpendicular bisector of $\overline{AA'}$.



- If a figure maps to itself under a reflection over a certain line, that line is called a **line of symmetry**.



- If we placed a ball at point Y and wanted to bounce it off $\overline{AB}$ to hit point X, then we would bounce it off point D, where D is the intersection between $\overline{AB}$ and $\overline{X'Y}$.

- For bouncing, we always minimize the distance of the path taken by the ball ($YD + DX$).

14.4 Dilation

- **Dilation** is a stretching (scale factor > 1) or shrinking (scale factor < 1) of a figure.
- The image of a point P upon dilation with **scale factor** k and **center of dilation** O is the point P' on $\overrightarrow{OP}$ such that $OP' = k(OP)$.[14]
  - In other words, $\frac{OP'}{OP} = k$, where $\overrightarrow{OA}$ passes through P'.
  - If your scale factor is negative, then P' goes in the opposite direction it usually would.
- A figure and its image upon dilation are similar. The ratio of corresponding sides of the figure and its image equals the scale factor of the dilation.
  - If A' and B' are the images of A and B, respectively, under a dilation with scale factor k, we have $\frac{A'B'}{AB} = k$.

14.5 General Ideas

- When solving for areas, you can often flip or slide the shape to make calculations easier.
- Consider extreme possibilities of problems.

15 Analytic Geometry

15.1 Lines

- We use the **Cartesian plane** to describe geometric figures with algebraic equations.[15]
- We call points that have integers for both coordinates **lattice points**.
- Line forms:

$$\text{Point-Slope Form: } y - y_1 = m(x - x_1)$$
$$\text{Standard Form: } Ax + By = C^{16}$$
$$\text{Slope-Intercept Form: } y = mx + b$$

- Distance Formula:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

[14] This applies when k is positive.

[15] Analytic geometry = geometry & algebra.

- Midpoint Formula:
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$
- Two lines are parallel if they have the *same* slope. Two lines are perpendicular if the product of their slopes is -1.

15.2 Circles

- The standard form of a circle with center (h, k) and radius r is
$$(x - h)^2 + (y - k)^2 = r^2.$$

15.3 Analytic Geometry

- Check if angles are $90°$.
- Try interpreting an equation as a geometric figure on a Cartesian plane.
- When setting up a geometric problem on a Cartesian plane, choose a convenient origin. Often, let the origin be the vertex of a right angle.
- Rectangles, midpoints, and right triangles hint at using analytic geometry.
- When rotating $x°$ around a point, use right triangles.
- When reflecting over a line, use the midpoint formula and slopes.
- Draw out diagrams and pay attention to points that are equidistant from certain lines.
- Remember that perpendicular slopes have a product of -1.
- When we see squares, we think of the distance formula.
- Tangents are closely related to perpendicular slopes.
- If a line bisects the area of a rectangle, it must pass through the rectangle's center.

15.4 Proofs with Analytic Geometry

- Choose the origin and coordinate axes wisely.

15.5 Distance Between a Point and a Line

- The distance between a point (x_0, y_0) and the graph of the equation $Ax + By + C = 0$ is
$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

16 Trigonometry

16.1 Intro to Trigonometry

- Important trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin\theta = \cos(90^\circ - \theta)$$
$$\cos\theta = \sin(90^\circ - \theta)$$
$$\sin(180^\circ - \theta) = \sin\theta$$
$$\cos(180^\circ - \theta) = -\cos\theta$$

16.2 Not Just for Right Triangles

- Use the unit circle to help find values of sin and cos.

16.3 Law of Sines and Cosines

- Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \frac{a^2+b^2-c^2}{2ab}$
- Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\text{R}$.

17 Problem Solving Strategies

17.1 The Extra Line

- Circles + tangent lines = draw a radius to a point of tangency.
- Draw segments that connect important points or those that form segments, triangles, or angles you know things about.
- 30°, 60°, and 120° are clues to build 30-60-90 triangles by dropping altitudes or extending segments.
- Segments that stop inside figures (triangles, quadrilaterals, or circles) are great candidates to be extended.
- Try using the same tactic multiple times in different ways if the first use hasn't solved the problem.
- When given circles and line segments, think of power of a point.

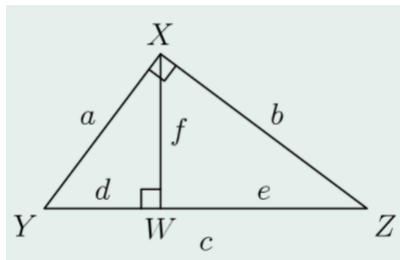
17.2 Assigning Variables

- Assign variables to convenient lengths/angles.
- We use the Pythagorean Theorem, similar triangles, and Power of a Point to chase lengths.
- If you need a length, draw altitudes and radii to points of tangency.

18 Proofs

18.1 Triangles

18.1.1 Pythagorean Theorem

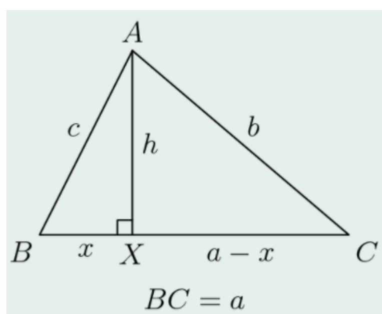


By $\triangle WXY \sim \triangle XZY$, $\frac{a}{c} = \frac{d}{a}$. Thus, $a^2 = cd$. By $\triangle WZX \sim \triangle XZY$, $\frac{b}{c} = \frac{e}{b}$. Thus, $b^2 = ce$. Therefore, if $\angle ZXY = 90°$,

$$a^2 + b^2 = cd + ce = c^2.$$

Note that the converse is true, too.[17] ■

18.1.2 Heron's Formula



Applying the Pythagorean Theorem to $\triangle ABX$ and $\triangle ACX$, we have

$$x^2 + h^2 = c^2, \tag{1}$$
$$(a - x)^2 + h^2 = b^2. \tag{2}$$

Subtracting (1) from (2), we have

$$(a - x)^2 - x^2 = b^2 - c^2.$$

[17]That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Solving for x, we arrive at

$$x = \frac{a^2 + c^2 - b^2}{2a}.$$

Factor out difference of squares from $h^2 = c^2 - x^2$ and plug in x:

$$\begin{aligned}
h^2 &= (c - x)(c + x) \\
&= (c - \frac{a^2 + c^2 - b^2}{2a})(c + \frac{a^2 + c^2 - b^2}{2a}) \\
&= (\frac{2ac - a^2 - c^2 + b^2}{2a})(\frac{2ac + a^2 + c^2 - b^2}{2a} \\
&= (\frac{b^2 - (a - c)^2}{2a})(\frac{(a + c)^2 - b^2}{2a}) \\
&= \frac{(-a + b + c)(a + b - c)(a - b + c)(a + b + c)}{4a^2}.
\end{aligned}$$

Define s to be the semiperimeter of $\triangle ABC$, where $s = \frac{a+b+c}{2}$. Then, $2s - 2a = -a + b + c$, $2s - 2b = a - b + c$, and $2s - 2c = a + b - c$. Plugging these in and manipulating, we get

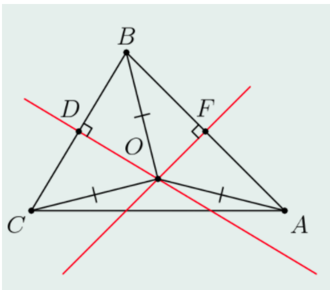
$$\begin{aligned}
h^2 &= \frac{(-a + b + c)(a + b - c)(a - b + c)(a + b + c)}{4a^2} \\
&= \frac{(2s - 2a)(2s - 2c)(2s - 2b)(2s)}{4a^2} \\
&= \frac{4s(s - a)(s - b)(s - c)}{a^2} \\
h &= \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}.
\end{aligned}$$

Using the formula $[ABC] = \frac{ah}{2}$, we arrive at

$$[ABC] = \sqrt{s(s - a)(s - b)(s - c)}.$$

■

18.1.3 Perpendicular Bisectors of a Triangle Meet at the Circumcenter



We ignore $\overline{EO}$ for a moment. Since $\overline{BC}$ and $\overline{AB}$ aren't parallel, they will intersect at point O.

Because O lies on the perpendicular bisector of $\overline{BC}$, we know it is equidistant from B and C. In other words, $OB = OC$. Following similar logic with the perpendicular bisector of $\overline{AB}$, we know that $OB = OA$.

Since $OA = OB = OC$, we know that O is equidistant from A and C. Thus, the perpendicular bisector of $\overline{AC}$ intersects point O. ■

18.1.4 The Midpoint of the Hypotenuse is the Circumcenter

18.1.5 All Points on an Angle Bisector are Equidistant from its Rays

18.1.6 Angle Bisectors of a Triangle Intersect at the Incenter

18.1.7 Angle Bisector Theorem

18.1.8 Medians of a Triangle Intersect at the Centroid

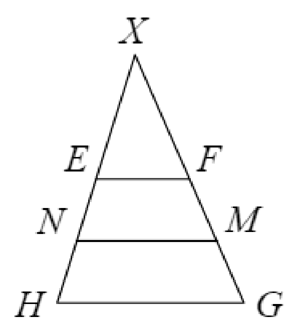
18.1.9 Medians Split a Triangle into Six Equal Areas

18.1.10 Medial Triangle Properties

18.1.11 Altitudes of a Triangle Intersect at the Orthocenter

18.2 Quadrilaterals

18.2.1 Median of a Trapezoid is Parallel to the Bases and the Average of Them



Given trapezoid $EFGH$, we extend $\overline{EH}$ and $\overline{FG}$ to meet at X. Because $\overline{EF} \parallel \overline{HG}$, we have that $\triangle EXF \sim \triangle HXG$.

Thus, $\frac{XE}{XH} = \frac{EF}{GH}$.

So, $\overline{EF} \parallel \overline{HG} \parallel \overline{NM}$.

Furthermore,

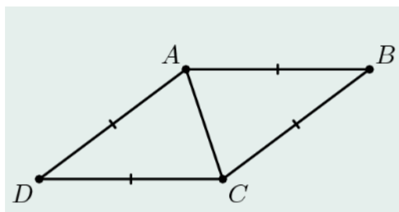
18.2.2 Isosceles Trapezoid Diagonals are Equal

18.2.3 Trapezoid Area Formula

18.2.4 Opposite Angles/Sides of a Parallelogram are Equal

18.2.5 Parallelogram Diagonals Bisect Each Other

18.2.6 Rhombus is a Parallelogram



Create diagonal $\overline{AC}$. Then, observe that $\triangle CDA \cong \triangle ABC$ by SSS.

We have $\angle DAC = \angle ACB$, which implies that $\overline{AD}$ and $\overline{BC}$ are parallel. Also, $\angle DCA = \angle BAC$, so $\overline{AB}$ and $\overline{DC}$ are parallel. This means $ABCD$ is a parallelogram. ■

18.2.7 Congruent segments iff congruent chords

18.3 Circles

18.3.1 Two Secants Intersecting