AP Physics C: Mechanics Summary

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Contents

*Many of these summaries are adapted from Princeton Review's AP Physics C Prep Book.

1 Notes

1.1 Introduction

This summary includes the main ideas of every unit in mechanics. This handout should act as a supplement to anything else you're studying, and not your main tool.

Equations that are on the equation sheet are boxed. Please use the equation sheet and this handout to determine which equations you should/shouldn't memorize.

1.2 Things to Memorize/Practice

1. Derivations for centers of mass
2. Derivations for rotational inertias
3. Common rotational inertias
4. Parallel-axis theorem

1.3 Contact Me

There may be some typos. If you notice any or have suggestions, please email michel.liao@systemgreen.org.

2 Kinematics

2.1 Definitions

- Position refers to an object relative to a coordinate axis.
- Distance refers to the *total* measure of the ground traveled by an object
- Displacement is how far an object is from where it started: $\Delta x = x_f - x_0$.
- Acceleration is a measure of change in velocity per some unit of time ($\overline{a} = \frac{\Delta v}{\Delta t}$). Acceleration is a *scalar*.

2.2 Uniformly Accelerated Motion

- There are 5 kinematics equations pertaining to uniformly accelerated motion (UAM)
  - $\Delta x = \overline{v}t$
  - $\boxed{v = v_o + at}$
  - $\boxed{x = x_0 + vt - \dfrac{1}{2}at^2}$
  - $\boxed{v^2 = v_0^2 + 2a(x - x_0)}$
  - $\overline{v} = \frac{1}{2}(v_0 + v)$

2.3 Non-uniformly Accelerated Motion

Unfortunately, this is where calculus comes in.

- Know your derivatives:
  - The derivative of $x(t)$ is velocity.
  - The derivative of $v(t)$ is acceleration.
- Know your integrals (remember $+C$):
  - The integral of $a(t)$ is velocity.
  - The integral of $v(t)$ is displacement.

2.4 Graphs

Know your integrals and derivatives as stated above. Use geometric figures to calculate areas.

2.5 Free Fall and Projectiles

- Free fall is when an object is only affected by the force of gravity.
- 2D motion should be analyzed according to its x and y components separately. The x and y components are independent.

3 Newton's Laws of Motion

3.1 Newton's Laws

Newton's Laws, in order, are:

1. Objects will continue in their state of motion unless acted upon by a net force.
2. $\boxed{\sum \vec{F} = m\vec{a}}$ [1]
3. When two objects interact, the force from the first object onto the second object is equal to, and in the opposite direction of, the force the second object exerts on the first object.

3.2 Weight, Normal Force, and Friction

- The weight of an object is *not* the mass of the object. Weight is given by $\vec{F_w} = m\vec{g}$.
- The normal force (F_N or N) is the component of the contact force exerted on an object that is perpendicular to the surface.
- Friction is the component of the contact force exerted on an object that is parallel to the surface.
- The inequality $\boxed{|\vec{F_f}| \leq \mu|\vec{F_N}|}$ gives you the *maximum* force that friction can apply [2] (see the footnote for more details).
  - Static friction occurs when there is no *relative* motion between the object and the surface.
  - Kinetic friction occurs when there is relative motion between the object and the surface.

3.3 Inclined Planes

- Rotate the x-y coordinate axis so that the x-axis and y-axis are parallel and perpendicular, respectively, to the incline (or else your life becomes way harder than it has to be).

[1] In your equation sheet, this is written as $\vec{a} = \frac{\sum \vec{F}}{m}$.

[2] In the case of static friction, we can imagine a box on a surface. The surface will only apply enough static friction force such that the net force remains $0N$. It doesn't make sense for the static friction force to always be equal to $\mu \cdot \vec{F_N}$, or the box may accelerate "backwards" when the forward force isn't equal to the maximum frictional force. Once a forward force exceeds the magnitude of maximum static friction force, the object will start sliding, and static friction becomes kinetic friction.

- There are two components of gravity that act on an object in an inclined plane: the force parallel to the ramp ($mg\sin\theta$) and the force perpendicular to the ramp ($mg\cos\theta$). [3]

3.4 Uniform Circular Motion

- An object undergoes uniform circular motion when its *speed* is constant.
- The velocity vector is *always* tangent to the circle.
- Centripetal acceleration and force points toward the center.

$$\boxed{a_c = \frac{v^2}{r}}$$

- Any force that points toward the center is positive, and any force that points away from the center is negative.
- When an object is at the top of a vertical circle, your acceleration <u>must</u> be at least $a = 9.8\frac{m}{s^2}$.
  - The proof is as follows: Assume we have a roller coaster at the top of a circular loop with mass m. For proof by contradiction, assume that $a < 9.8\frac{m}{s^2}$. Then, we have

$$\begin{aligned} F_N + F_g &= ma \\ F_N &= ma - F_g = m(a - g) \\ F_N &< 0N \end{aligned}$$

    Which is clearly impossible. A normal force that is $0N$ implies that it has no contact with the surface. Therefore, a must be at least $9.8\frac{m}{s^2}$ at the top of a vertical circle. [4]

[3] I recommend that you don't memorize these formulas. Rather, know the definitions of trigonometric functions (SOH CAH TOA).

[4] Remember that acceleration is a vector. The speed of our object does not change, but the direction does.

4 Work, Energy, and Power

4.1 Work

- Work is the dot product of force and displacement: $W = F \cdot x$.
- If work isn't constant, then $\boxed{W = \int F(x)dx}$.
- Work is positive if the force is in the same direction of displacement. Conversely, work is negative if the force is opposite the direction of displacement.
- The total work done on an object/system is the sum of all work done by every force on the object/system: $W_{\text{total}} = \sum W$.
- The Work-Energy Theorem: $W_{\text{total}} = \Delta K$. A corollary is $\Delta U_g = -W_{\text{by gravity}}$.[5]

4.2 Energy

- Kinetic energy is energy associated with motion: $\boxed{K = \frac{1}{2}mv^2}$.
- Potential energy is stored energy, or the ability of "potentially" moving:[6]
  - $\boxed{\Delta U_g = mg\Delta h}$.
  - $\boxed{U_s = \frac{1}{2}k(\Delta x)^2}$.
  - You cannot have potential energy if you do not include the Earth in your system.[7]
- A conservative force is defined by having the total work done by moving between two points as independent of the path taken (e.g., F_g, F_s).
- Work done by a non-conservative force (e.g., friction and air resistance) depends on the path taken, and mechanical energy is lost by heat, sound, etc.
- Conservation of Mechanical Energy occurs when there are no non-conservative forces acting on the system. There are several ways to express conservation of energy:
  - $E_0 = E_f \Rightarrow K_0 + U_0 = K_f + U_f$.

[5] The distinction between W_{total} and $W_{\text{by gravity}}$ is small, but important.

[6] Potential energy is relative. To make your life easier, choose strategic inital potential energy positions.

[7] The statement isn't exactly correct, but it's a general way to think about things. If you don't include the Earth in your system, then F_g is an external force, which you would add into the left side of your equation (look at the conservation of energy section).

– $\Delta K = -\Delta U \Rightarrow \Delta K + \Delta U = 0J$.
– If there are non-conservative forces that do work, then:

$$K_0 + U_0 + W_{\text{other}} = K_f + U_f.$$

4.3 Potential Energy Diagrams

- Potential energy is given as $U(x)$. Then, $F = -\frac{dU}{dx}$.
- If $\frac{dU}{dx} = 0$, then $F = 0$ and it is an equilibrium point.[8]
  - Stable equilibrium occurs when the force restores the object back toward equilibrium after it is disturbed.
  - Unstable equilibrium occurs when the force moves the object further away from the equilibrium point after it is disturbed.

4.4 Power

- Power is the rate at which work is done.[9]
  - $$\boxed{P = \frac{W}{t} = \frac{dW}{dt}}$$
  - $$\boxed{P = \vec{F} \cdot \vec{v}}$$ [10]

[8] On an energy vs. position graph, think of the graph as a hill. Place a "ball" at some point, and if it will roll to a point and stay there, that is a point of stable equilibrium. The opposite is true, too.

[9] The equation sheet has the first equation written as $P = \frac{dE}{dt}$.

[10] This equation involves the dot product. Your velocity must be parallel to your force. (or vice versa).

5 Systems of Particles and Linear Momentum

5.1 Momentum

- Linear momentum is given by the equation

$$\boxed{\vec{p} = m\vec{v}}.$$

- We can derive momentum with respect to time to get force:

$$\boxed{\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}}.$$

- The Law of Conservation of Momentum says that linear momentum is conserved who no external forces act on the system, or $p_0 = p$.
- Elastic collisions conserve *kinetic energy.*[11]
- Generally, for elastic collisions, the relative velocities $(v_2 - v_1)$ after the collision is equal in magnitude and opposite to the relative velocity before the collision $(v_{02} - v_{01})$.
- Inelastic collisions do not conserve kinetic energy. When objects stick together post-collision, the collision is said to be *perfectly* inelastic.

5.2 Impulse

- Impulse is given by the equation

$$\boxed{\vec{J} = \bar{F}\Delta t = \int \vec{F} dt = \Delta\vec{p}}^{12}.$$

5.3 Center of Mass

- Newton's Second Law, $\sum \vec{F} = m\vec{a}$, calculates the acceleration of the center of mass of your system. If the net external force on your system is 0 N, then your center of mass does not accelerate.
- For a uniform density object, the center of mass is at its geometric center.
- For point masses, the center of mass is given by

$$\boxed{x_{\text{cm}} = \frac{\sum m_i x_i}{\sum m_i}}.$$

[11] Kinetic energy is the key term. In general, every collision conserves energy (energy turns into heat, sound, etc.), but not kinetic energy.

[12] $\bar{F}\Delta t$ is not in your equation sheet, but you should memorize it

- For non-uniform densities, the center of mass is given by

$$x_{\text{cm}} = \frac{1}{m_{\text{total}}} \int r\, dm.$$

where M is your total mass.[13]

  - Use the definition of linear mass density ($\lambda = \frac{dm}{dx} = \frac{M}{L}$) to get rid of dm.

[13] This equation isn't on your equation sheet, but you should memorize it.

6 Rotation

6.1 Linear and Angular Quantities

- $s = r\theta$
- $v = r\omega$[14]
- $a_t = r\alpha$[15]

Linear Variables	Angular Equivalent
x	θ
v	ω
a	α
p	L
m	I
F	τ

6.2 Basic Rotation Information

- Rotational inertia is a measure of how difficult it is to change an object's rotational motion.

$$\boxed{I = mr^2 = \int r^2 dm}$$

- The Parallel-Axis Theorem applies when the object is uniform

$$I = I_{\text{cm}} + mx^2.$$

- Torque is the ability to cause an object to rotate:

$$\boxed{\tau = r \times F}.$$

- Newton's Second Law's rotational equivalent is

$$\boxed{\sum \tau = I\alpha}.$$

- Rotating objects have rotational kinetic energy given by

$$\boxed{K_{\text{rotation}} = \frac{1}{2}I\omega^2}.$$

[14] This velocity is tangential velocity. This is sort of redundant to say, as all velocity must be tangential in a curving path, but it is important to know that centripetal velocity doesn't exist.

[15] This equation relates to *tangential* acceleration, not centripetal. To find the magnitude of total acceleration, use the equation $a = \sqrt{a_t^2 + a_c^2}$.

- If an object is rolling, the rotational kinetic energy is

$$K_{\text{rolling}} = K_{\text{rotation}} + K_{\text{translation}} = \frac{1}{2}I_{\text{cm}}\omega^2 + \frac{1}{2}mv_{\text{cm}}^2.$$

[16]

6.3 Angular Momentum

- Angular momentum for a point particle is given by the equation

$$\boxed{L = I\omega}.$$

- Angular momentum for a rigid object is given by

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}.$$ [17]

- Angular momentum is conserved unless a net external torque acts on the object. We use the equation

$$\sum \tau = \frac{dL}{dt}.$$ [18]

- Know your different types of equilibrium:
  - Translational equilibrium is when the net force is $0N$.
  - Rotational equilibrium occurs when the net torque is $0\frac{kg \cdot m^2}{s^2}$.
  - Equilibrium (by itself) occurs when there is both translational and rotational equilibrium.[19]
  - If an ob ject is at rest, it is in static equilibrium (both translational and rotational).

[16] Note the emphasis put on the center of mass.

[17] This is how we find angular momentum in terms of linear momentum.

[18] This equation isn't on your equation sheet, but don't worry! $F = \frac{dp}{dt}$ is, and you can just use your rotational equivalents.

[19] Remember that when we're in either translational or rotational equilibrium, it doesn't necessarily mean that our velocity or angular velocity is 0. It just means that our velocity/angular velocity is <u>not</u> changing.

7 Oscillations

7.1 Simple Harmonic Motion

- Simple harmonic occurs when there is a restoring force on an object that is proportional for the displacement. The restoring force for a spring is

$$\boxed{F_s = -kx}.$$

[20]

- The equation for an object undergoing SHM is

$$\boxed{x = x_{\text{max}} \cos(\omega t + \phi)}.$$

- The period is the length of time it takes to copmlete one cycle and frequency is the number of cycles the object completes in one unit of time:

$$\boxed{f = \frac{1}{T} = \frac{\text{cycles}}{\text{time}}}.$$

- Differential equations are cool. We have two different cases, namely springs and pendulums.[21]
  - Springs: always start with $F = ma$ and replace F.

$$\begin{aligned} F &= ma \\ -kx &= m\frac{d^2x}{dt^2} \\ -\frac{k}{m}x &= \frac{d^2x}{dt^2}. \end{aligned}$$

[20] Remember, x is the length the spring will stretch beyond its natural length (nothing attached to it).

[21] You don't actually *have* to start with $F = ma$, but it's general rule. Sometimes, start with $\tau = I\alpha$.

– Pendulums: always start with $F = ma$ and replace F.[22]

$$\begin{aligned}
F &= ma \\
-mg\sin\theta &= m\frac{d^2s}{dt^2} \\
-g\theta &= \frac{d^2s}{dt^2} \\
-g\theta &= L\frac{d^2\theta}{dt^2} \\
-\frac{g}{L}\theta &= \frac{d^2\theta}{dt^2}.
\end{aligned}$$

[22]This is why small angle approximations are so key to SHM with pendulums—so we can assume $\sin\theta = \theta$. Also, note that because $s = L\theta$, $\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}$.

8 Gravitation

8.1 Kepler's Laws

In order, Kepler's Laws are:

1. Every planet moves in an elliptical orbit, with the Sun at one focus.[23]
2. A line draw from the Sun to the planet sweeps out equal areas in equal time intervals, or conservation of angular momentum.
3. If T is the period and a is the length of the semimajor axis of a planet's orbit, then
$$\frac{T^2}{a^3}$$
if the orbit is elliptical, and
$$\frac{T^2}{R^3}$$
if the planet's orbit is circular.[24]

8.2 Newton's Law of Gravitation & Circular Orbits

- Newton's Law of Gravitation:
$$\boxed{F_g = \frac{Gm_1 m_2}{r^2}}.$$
Use it in conjunction with $F_g = ma_g$ to arrive at
$$a_g = \frac{Gm_1}{r^2}.$$
- $F_g = F_c$ is very helpful to find the speed of an orbiting object.
- $$v = \frac{2\pi r}{T}$$

8.3 General Orbits

- Gravitational potential energy is given by
$$\boxed{U_g = -\frac{Gm_1 m_2}{r}}.$$

[23] This is technically wrong. The planet and the Sun will orbit around the planet-Sun system's center of mass. However, because the distance from the center of mass to the Sun is so small, we approximate the center of mass to be at the position of the Sun.

[24] R represents the radius of the circular orbit.

- Mechanical energy and angular moment are conserved for orbits.
- Escape speed is derived when we set our kinetic and potential energy equal to 0.

  $$\frac{1}{2}m_1 v_{\mathrm{esc}^2} - \frac{Gm_1m_2}{r} = 0$$

  Which gives us

  $$v_{\mathrm{esc}} = \sqrt{\frac{2Gm_2}{r}}.$$

- The total energy of a satellite in a circular orbit of radius R is

  $$E = -\frac{GMm}{2R}.$$

  The total energy of a satellite in a elliptical orbit if the semimajor axis is a is

  $$E = -\frac{GMm}{2a}.$$

8.4 Gravity of Spheres and Shells

- The force gravity outside a sphere is

  $$F_g = \frac{GMm}{r^2}.$$

  The force gravity inside a sphere is

  $$F_g = \frac{GMm}{R^3}r.$$